THE PROPAGATION OF LONG ELECTRIC WAVES ALONG WIRES.

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(*Read December 2, 1910.*)

In the usual deduction of the equations of propagation of electric waves along wires the notion of the electric constants per unit length is introduced. While there is no difficulty involved in this as far as resistance and leakage are concerned, the legitimacy of the extension of this notion to self-induction and capacity is not obvious. In order to determine the exact meaning to be attached to these terms it is convenient to consider a line in which the electric properties are localized in a finite number of coils, condensers and leaks, joined by ideal conductors of no resistance, self-induction and capacity. For the special case of long electric waves the solution can readily be obtained by means of the calculus of finite differences. On passing to the limit, by letting the number of coils, etc., increase indefinitely while their electric constants decrease indefinitely, the equations of propagation and their solution for a uniform line are at once obtained. There appears to be a considerable advantage in the use of this method in respect to its simplicity, particularly where the terminal conditions are at all complicated. Two problems are worked out in this paper; the first that of the free vibrations of a line earthed at both ends, and the second that of the forced vibrations when a periodic impressed electromotive force is applied to the circuit.

Consider a line of length l, in which are inserted at equal intervals n coils each of resistance R' and self-induction L. At points between each pair of coils one plate of a condenser of capacity S is connected, the other plate being earthed; and at the same points leaks to earth, each of conductance K', are introduced. The current in the kth coil is C_k, and the potential at a point between the coils k and $k+1$ is V_k. For the first problem we then have $V_0 = V_n = 0$. Let L_1 be the coefficient of mutual induction between any coil and one of its nearest neighbors; L_2 the coefficient of mutual induction between the same coil and its next nearest neighbor but one, and so on. Similarly, let S_1 be the electric induction coefficient between any condenser and one of its nearest neighbors; S_2 the induction coefficient between two alternate condensers, etc. We can then write for the kth coil:

$$V_{k-1} - V_k = \frac{d}{dt}\Big(LC_k + L_1 C_{k-1} + L_1 C_{k+1} + L_2 C_{k-2} + L_2 C_{k+2} + \cdots\Big) + R'C_k \tag{1}$$

and for the kth condenser:

$$C_k - C_{k+1} = \frac{d}{dt}\Big(SV_k + S_1 V_{k-1} + S_1 V_{k+1} + S_2 V_{k-2} + S_2 V_{k+2} + \cdots\Big) + K V_k \tag{2}$$

Now in the case of long electric waves the currents in any coil and its near neighbors will be very nearly the same. The terms in the series in (1) containing currents in distant coils become relatively unimportant on account of the diminution of their coefficients. In this special case it will therefore be legitimate to replace the series by a single term and we can therefore write:

$$V_{k+1} - V_k = L' \frac{d}{dt} C_k + R' C_k, \tag{3}$$

in which L' may be termed the effective coefficient of self-induction of any one of the coils. When we pass to the limit by increasing indefinitely the number of coils, etc., and at the same time decreasing indefinitely all the electric constants, the limiting value which the product of L' by the number of coils in a unit length approaches will be the self-induction per unit length of the uniform line. Equation (2) modified in an analogous manner reduces to:

$$C_k - C_{k+1} = S' \frac{d}{dt} V_k + K' V_k \tag{4}$$

Subtracting the equation for the coil $k+1$ from (3) and substituting from (4), we get:

$$V_{k-1}-(2+h)V_k+V_{k+1}=0, \tag{5}$$

where

$$h=R'K'-L'S'p^2+ip(L'K'+R'S'), \tag{6}$$

in which it is assumed that the potentials and currents all vary as e^{ipt}. (5) is a linear difference equation of the second order. By the usual method we put

$$V_k=A_k\alpha^k e^{ipt},$$

and find:

$$\alpha=\frac{2+h}{2}\pm\tfrac{1}{2}\sqrt{4h+h^2}. \tag{7}$$

Let these two values be α and β, the former with the positive sign of the radical. In general α and β are different, and so we get the two distinct solutions required by an equation of the second order. But for $h=0$ or $h=-4$ α and β have the same values. The complete solution of (5) is therefore:

$$V_k=(A_1+B_1k)e^{ip_1t}+(A_2+B_2k)e^{ip_2t}+(A_3+B_3k)(-1)e^{ip_3t}$$
$$+(A_4+B_4k)(-1)^k e^{ip_4t}+\sum(A_p\alpha^k+B_p\beta^k)e^{ipt}.$$

Since $V_0=V_n=0$, whatever t, $A_1=A_2=A_3=A_4=B_1=B_2=B_3=B_4=0$; $A_p+B_p=0$ and

$$\alpha^n-\beta^n=0. \tag{8}$$

Let now

$$h=-4\sin^2\theta, \tag{9}$$

(7) reduces to

$$\alpha,\beta=\cos 2\theta\pm i\sin 2\theta, \tag{10}$$

and (8) gives

$$\theta=\frac{m\pi}{2n}, \tag{11}$$

where m is any integer. $m=0$ and $m=n$ are excluded because these give $h=0$ and $h=-4$, which are already disposed of. If we take $m=n+1$, $n+2$, etc., we get the same series of values obtained from $m=1$ to $m=n-1$. We thus get:

$$V_k = \sum_{m=1}^{m=n-1} A_m \sin \frac{km\pi}{n} e^{ip_m t}. \tag{12}$$

A_m and p_m are complex quantities. Writing $p_m = p_m' + ip_m''$, we get from (6), (9) and (10):

$$p_m' = \sqrt{\frac{4\sin^2\dfrac{m\pi}{2n}}{L'S'} - \frac{(L'K' - R'S')^2}{4L'^2S'^2}}, \tag{13}$$

$$p_m'' = \frac{L'K' + R'S'}{2L'S'} = p''. \tag{14}$$

p_m'' is thus independent of m. The real part of (12) may now be written

$$V_k = e^{-p''t} \sum_1^{n-1} A_m \cos(p_m't - \phi_m) \sin \frac{km\pi}{n}, \tag{15}$$

where A_m and ϕ_m are new arbitrary real constants.

The currents in the several coils may be obtained from (3) combined with (12). Taking the real part we find

$$C_k = e^{-p''t} \sum_1^{n-1} B_m \cos(p_m't - \psi_m) \cos(2k-1)\frac{m\pi}{n} \tag{16}$$

where B_m and ψ_m are known in terms of A_m and ϕ_m in (15).

The last four equations give the complete solution of the problem. The constants A_m and ϕ_m may be determined by Fourier's method when the initial conditions are known.

Now let n increase indefinitely while R', L', S' and K' all decrease indefinitely. Let $L = \text{limit } L'n/l$, and similarly for the others. Let δx be the distance between two coils, so that $n\delta x = l$. Measuring x from the end of the line corresponding to $k=0$, we have $k = nx/l$ and we get in the limit:

$$V = e^{-p''t} \sum_1^{\infty} A_m \cos(p_m't - \phi_m) \sin \frac{m\pi x}{l}, \tag{17}$$

$$C = e^{-p''t} \sum_1^{\infty} B_m \cos(p'_m t - \psi_m) \cos \frac{m\pi x}{l}, \tag{18}$$

$$p_m' = \sqrt{\frac{m^2\pi^2}{l^2LS} - \left(\frac{LK - RS}{2LS}\right)^2}, \tag{19}$$

$$p'' = \frac{LK + RS}{2LS}, \tag{20}$$

which are the well-known solutions for the free vibrations of a uniform line for long waves.

The differential equation of which (17) is the solution is obtained by passing to the limit in the difference equation (5). We thus get

$$\frac{\partial^2 V}{\partial x^2} = LS\frac{\partial^2 V}{\partial t^2} + (LK + RS)\frac{\partial V}{\partial t} + RKV.$$

Equations (3) and (4) on passing to the limit give:

$$-\frac{\partial V}{\partial x} = L\frac{\partial C}{\partial t} + RC,$$

$$-\frac{\partial C}{\partial x} = S\frac{\partial V}{\partial t} + KV.$$

For the second problem, that of a periodic impressed electromotive force applied to one end of a line, the other end being earthed, we have to solve equation (5) subject to the conditions:

$$\begin{aligned} k &= 0, \quad V_k = Ee^{i\nu t}, \\ k &= n, \quad V_k = 0. \end{aligned} \tag{21}$$

The resulting solution may of course be applied to a closed circuit with the periodic force $Ee^{i\nu t}$ introduced in it at any point. After the free vibrations have been damped out, the solution will be

$$V_k = (A\alpha^k + B\beta^k)Ee^{i\nu t},$$

where A and B are arbitrary constants and α and β are given by (10). Determining A and B by means of (21) we get

$$V_k = \frac{\sin 2(n-k)\theta}{\sin 2n\theta} Ee^{i\nu t}. \tag{22}$$

θ is a complex angle defined by (6) and (9) if ν is written for p

in (6). Putting $\theta = \theta' + i\theta''$, we get as the real part of (22)

$$V_k = \frac{E}{2(\cosh 4n\theta'' - \cos 4n\theta')}\{e^{2(2n-k)\theta''}\cos(\nu t + 2k\theta') + e^{-2(2n-k)\theta''}\cos(\nu t - 2k\theta') - e^{2k\theta''}\cos(\nu t + 4n\theta' - 2k\theta') - e^{-2k\theta''}\cos(\nu t - 4n\theta' + 2k\theta')\}, \tag{23}$$

which together with

$$4(\sin^2\theta' \cosh^2\theta'' - \cos^2\theta' \sinh^2\theta'') = \nu^2 L'S' - R'K' \tag{24}$$

$$-4\sin 2\theta' \sinh 2\theta'' = \nu(L'K' + R'S') \tag{25}$$

gives the complete solution.

Now on passing to the limit as before, we can replace $\sin\theta$ by θ, $L' = L\delta x$, etc., and we find

$$2\theta' = -Q\delta x,$$
$$2\theta'' = P\delta x,$$

where

$$P, Q = \frac{1}{\sqrt{2}}\{(\nu^2L^2 + R^2)(\nu^2S^2 + K^2) \pm (RK - \nu^2LS)\}^{\frac{1}{2}},$$

We thus have

$$4n\theta'' = 2Pl, \quad 4n\theta' = -2Ql,$$
$$2k\theta'' = Px, \quad 2k\theta' = -Qx;$$

(23) thus reduces to

$$V = Ee^{-Px}\cos(\nu t - Qx) + \frac{Ee^{-Pl}}{2(\cosh 2Pl - \cos 2Ql)^{\frac{1}{2}}} \times \{e^{Px}\cos(\nu t + Qx + \phi) - e^{-Px}\cos(\nu t - Qx + \phi)\},$$

where

$$\tan\phi = \frac{\sin 2Ql}{e^{-2Pl} - \cos 2Ql},$$

which is the solution for this case as given by Heaviside,[1] except that leakage is here considered and the real impressed force is $E\cos\nu t$ instead of $E\sin\nu t$.

[1] "Electrical Papers," Vol. 2, p. 62.

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